

$$\begin{aligned} 1a \quad \mathcal{L}\{te^{5t}\}(s) &= \int_0^{\infty} te^{5t} e^{-st} dt \\ &= \int_0^{\infty} te^{(5-s)t} dt \\ &= \underbrace{\frac{te^{(5-s)t}}{5-s}}_{t=0} \Big|_{t=0}^{\infty} - \int_0^{\infty} \frac{e^{(5-s)t}}{5-s} dt \end{aligned}$$

$t=\infty$ term only vanishes if $\operatorname{Re}(s) > 5$

$$\begin{aligned} &= 0 - \frac{1}{5-s} \int_0^{\infty} e^{(5-s)t} dt \\ &= -\frac{1}{5-s} \frac{e^{(5-s)t}}{5-s} \Big|_{t=0}^{\infty} \\ &= \begin{cases} \frac{1}{(5-s)^2} & \text{if } \operatorname{Re}(s) > 5 \\ \text{undefined} & \text{if } \operatorname{Re}(s) \leq 5 \end{cases} \end{aligned}$$

1b The domain of this function is $\{s \in \mathbb{C} : \operatorname{Re}(s) > 5\}$.

An answer of $s > 5$ is also considered correct. (If you want to think of $\mathcal{L}\{te^{5t}\}$ as a real-valued function.)

$$2 \quad V(t) = L \frac{d^2 Q}{dt^2} + \frac{Q}{C} ; \quad V(t) = \delta_1(t)$$

Take L.T. of both sides:

$$L\{\delta_1(t)\}(s) = L\{d^2 Q/dt^2\} + C^{-1} L\{Q\}(s)$$

$$e^{-s} = L(-Q'(0) - sQ(0) + s^2 L\{Q\}(s)) + C^{-1} L\{Q\}(s)$$

$$L\{Q\}(s) = \frac{e^{-s} + L(Q'(0)) + LsQ(0)}{Ls^2 + 1/C}$$

$$Q(t) = L^{-1}\left\{\frac{e^{-s}}{Ls^2 + 1/C}\right\} + L^{-1}\left\{\frac{Q'(0) + sQ(0)}{s^2 + 1/LC}\right\}$$

Calculate each separately using a table.

$$\begin{aligned} L^{-1}\left\{\frac{Q'(0)}{s^2 + 1/LC}\right\} &= Q'(0)\sqrt{LC} \, L^{-1}\left\{\frac{\sqrt{1/LC}}{s^2 + (\sqrt{1/LC})^2}\right\} \\ &= Q'(0)\sqrt{LC} \sin(\sqrt{1/LC} \, t) \end{aligned}$$

$$L^{-1}\left\{\frac{sQ(0)}{s^2 + 1/LC}\right\} = Q(0) \cos(\sqrt{1/LC} \, t)$$

For the next part we need to use that the inverse L.T. of a product is the convolution of inverse L.T.'s:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F\}u_0(t) * \mathcal{L}^{-1}\{G\}u_0(t)$$

$$\mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{ls^2 + 1/c}\right\} = u_0(t) \mathcal{L}^{-1}\{e^{-s}\} * u_0(t) \mathcal{L}^{-1}\left\{\frac{1}{ls^2 + 1/c}\right\}$$

$$= u_0(t) \delta_1(t) * u_0(t) \frac{\sqrt{lc}}{l} \mathcal{L}^{-1}\left\{\frac{1/\sqrt{lc}}{s^2 + (\sqrt{l/c})^2}\right\}$$

$$= \delta_1(t) * u_0(t) \sqrt{c/l} \sin(t/\sqrt{lc})$$

$$= \int_{-\infty}^{\infty} \frac{u_0(\tau-t)}{\sqrt{l/c}} \sin((\tau-t)/\sqrt{lc}) \delta_1(\tau) d\tau$$

$$= \frac{u_0(1-t)}{\sqrt{l/c}} \sin((1-t)/\sqrt{lc})$$

$$\nwarrow u_0(1-t) = \begin{array}{c} \text{---} \\ | \\ t=1 \end{array}$$

Hence

$$= 1 - u_1(t)$$

$$\begin{aligned} Q(t) &= Q'(0) \sqrt{lc} \sin(t/\sqrt{lc}) \\ &\quad + Q(0) \cos(t/\sqrt{lc}) \\ &\quad + \frac{1 - u_1(t)}{\sqrt{l/c}} \sin((1-t)/\sqrt{lc}) \end{aligned}$$

$$3 \quad x(k) = \begin{cases} 1 & \text{if } k > 0 \text{ is even} \\ 0 & \text{else} \end{cases}$$

$$y(k) = \begin{cases} 3^k & \text{if } k < 0 \\ 0 & \text{else} \end{cases}$$

$$Z\{x\}(z) = \sum_{k \geq 0} \begin{pmatrix} 1 & \text{if } k \text{ even} \\ 0 & \text{else} \end{pmatrix} z^k$$

$$= \sum_{m \geq 0} z^{2m}$$

$$= \sum_{m \geq 0} (z^2)^m$$

$$= \frac{1}{1 - z^2}$$

$$(|z| < 1)$$

$$Z\{y\}(z) = \sum_{k=-\infty}^{-1} 3^k z^k$$

$$= \sum_{m=1}^{\infty} (3z)^{-m} \quad (m = -k)$$

$$= \sum_{n=0}^{\infty} (3z)^{-n-1} \quad (n = m-1)$$

$$= \frac{1}{3z} \sum_{n=0}^{\infty} \left(\frac{1}{3z} \right)^n$$

$$= \frac{1}{3z} \frac{1}{1 - 1/3z}$$

$$(|1/3z| < 1)$$

$$\Leftrightarrow |z| > 1/3$$

4 a $f(z) = \frac{1}{z^2 + 1}$, Region of conv. is $|z| < 1$

On this region, $|-z^2| < 1$, so

$$\frac{1}{1 - (-z^2)} = \sum_{k \geq 0} (-z^2)^k$$

$$= \sum_{k \geq 0} (-1)^k z^{2k}$$

$$= \sum_{m \geq 0} \left(\begin{cases} (-1)^{m/2} & \text{if } m \text{ even} \\ 0 & \text{else} \end{cases} \right) z^m$$

So $z^{-1} \{f\}(m) = \begin{cases} (-1)^{m/2} & \text{if } m \geq 0 \text{ is even} \\ 0 & \text{else} \end{cases}$

4b Now region of conv. is $|z| > 1$

$$\frac{1}{z^2 + 1} = \frac{1}{z^2} \frac{1}{1 - (-1/z^2)}$$

Since $|z| > 1$, it follows that

$$|-1/z^2| < 1.$$

So $\frac{1}{1 - (-1/z^2)} = \sum_{k \geq 0} \left(\frac{-1}{z^2} \right)^k$ on $|z| > 1$

$$\frac{1}{z^2 + 1} = \frac{1}{z^2} \sum_{k \geq 0} (-1)^k z^{-2k}$$

$$= \sum_{k \geq 0} (-1)^k z^{-2(k+1)}$$

$$k+1 = -m$$

$$= \sum_{m=-\infty}^{-1} (-1)^{-1-m} z^{-2m}$$

$z^{-1} \{f\}(k) = \begin{cases} (-1)^{-1-k/2} & \text{if } m < 0 \text{ is even} \\ 0 & \text{else.} \end{cases}$